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SHARP DISTORTION THEOREMS FOR QUASICONFORMAL MAPPINGS

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ABSTRACT. Continuing their earlier work on distortion theory, the authors prove some dimension-free distortion theorems for K-quasiconformal mappings in R^n. For example, one of the present results is the following sharp variant of the Schwarz lemma: If f is a K-quasiconformal self-mapping of the unit ball B^n, $n \geqslant 2$, with $f(0) = 0$, then $4^{1-K^2}|x|^K \leqslant |f(x)| \leqslant 4^{1-1/K^2}|x|^{1/K}$ for all x in B^n.

1. Introduction. Some fundamental distortion properties of K-quasiconformal mappings of the unit ball B^n in R^n are expressed by the two facts that these mappings (1) are Hölder continuous and (2) satisfy a generalized Schwarz lemma. Such results were first obtained in the plane case by L. V. Ahlfors [**Ah1**] and J. Hersch and A. Pfluger [**HP**]. Later the multidimensional case was studied by E. D. Callender [**C**], F. W. Gehring [**G**], Yu. G. Reshetnyak [**R1, R2**], B. V. Shabat [**Sh**], and O. Martio, S. Rickman, and J. Väisälä [**MRV**]. These n-dimensional results depend essentially on n and they contain constants that are unbounded as n tends to ∞. The present authors [**AVV**] recently improved these results by showing that there also exists a dimension-free distortion theory of K-quasiconformal mappings in R^n.

In the present paper we continue our earlier work and examine two conjectures due to A. V. Sychev and to T. Iwaniec, respectively. After this we prove a dimension-free variant of a distortion theorem due to F. W. Gehring and B. G. Osgood [**GO**, Theorem 3].

Before stating the main results of this paper we introduce some necessary notation and recall those well-known results which will be used throughout. The Grötzsch and Teichmüller condensers in R^n are denoted by $R_{G,n}(s)$, $s > 1$, and $R_{T,n}(t)$, $t > 0$, respectively. Their (conformal) capacities are denoted as in [**AVV**] by

$$\gamma(s) = \gamma_n(s) = \operatorname{cap} R_{G,n}(s), \qquad s > 1,$$
$$\tau(t) = \tau_n(t) = \operatorname{cap} R_{T,n}(t), \qquad t > 0.$$

These functions satisfy the basic functional identity

$$\gamma(s) = 2^{n-1}\tau(s^2 - 1), \qquad s > 1. \tag{1.1}$$

We sometimes omit the subscript n if there is no danger of confusion.

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Next, for $K > 0$ and $n \geqslant 2$, we define a homeomorphism $\varphi_K = \varphi_{K,n}\colon [0,1] \to [0,1]$ by $\varphi_K(0) = 0$, $\varphi_K(1) = 1$ and

(1.2) $$\varphi_K(r) = \varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/r))}$$

when $0 < r < 1$. For $K \geqslant 1$ and $n \geqslant 2$, we also set

(1.3) $$\varphi_K^*(r) = \varphi_{K,n}^*(r) = \sup\{|f(x)|\colon f \in QC_K(B^n),\, f(0) = 0,\, |x| = r\},$$

$r \in [0,1)$, and $\varphi_{K,n}^*(1) = 1$, where $QC_K(B^n)$ is the set of all K-quasiconformal mappings f of B^n into B^n. By the quasiconformal Schwarz lemma [**Sh**; **MRV**, 3.1; **W**] we obtain

(1.4) $$\varphi_{K,n}^*(r) \leqslant \varphi_{K,n}(r)$$

for all $K \geqslant 1$, $n \geqslant 2$, and $r \in [0,1]$. By [**LV**, p. 64, Theorem 3.1],

(1.5) $$\varphi_{K,2}^*(r) = \varphi_{K,2}(r)$$

for all $K \geqslant 1$ and $r \in [0,1]$. Whether the n-dimensional analogue of (1.5) holds for $n \geqslant 3$ is an open problem. By [**W**; **AVV**, 4.10, 4.14], for all $n \geqslant 2$ and $K \geqslant 1$,

(1.6) $$r^\alpha \leqslant \varphi_{K,n}(r) \leqslant \lambda_n^{1-\alpha} r^\alpha \leqslant 2^{1-1/K} K r^\alpha, \qquad \alpha = K^{1/(1-n)},$$

where $\lambda_n \in [4, 2e^{n-1})$ [**An**, (4)] is the Grötzsch ring constant depending only on n. The weaker bound $\varphi_{K,n}(r) \leqslant \lambda_n r^\alpha$ is given in [**MRV**, 2.8].

It was conjectured by A. V. Sychev [**Sy**, p. 89, Remark 2] that

(1.7) $$\varphi_{K,n}^*(r) \leqslant 4^{1-\alpha} r^\alpha, \qquad \alpha = K^{1/(1-n)},$$

for all $n \geqslant 3$ and $K \geqslant 1$. Since $\lambda_2 = 4$ [**LV**, p. 61], inequality (1.7) follows from (1.6) [**W**] when $n = 2$. But since $\lim_n \lambda_n = \infty$ [**An**], one cannot derive (1.7) from (1.6) and (1.4) for general n. Furthermore, it follows from [**AVV**], as we shall point out below in Remark 2.28, that (1.7) with $\varphi_{K,n}^*$ replaced by $\varphi_{K,n}$ is false. However, by using Lemma 2.3 below (an improved version of [**Vu2**, 3.5]) along with (1.6) and (1.4) we are able to prove our first theorem, which is very close to Sychev's conjecture.

1.8. THEOREM. *For $n \geqslant 2$, $K \geqslant 1$, and $r \in [0,1]$ the following inequality holds*:

(1.9) $$\varphi_{K,n}^*(r) \leqslant 4^{1-1/K^2} r^{1/K}.$$

This inequality is sharp when $K = 1$. Furthermore, the constant 4 cannot be replaced by a smaller one independent of K.

Our second theorem gives an affirmative answer to a conjecture communicated by T. Iwaniec to M. Vuorinen during the 1978 ICM in Helsinki. The theorem is applicable to a class of mappings that is larger than $QC_K(B^n)$, namely the class $QR_K(B^n)$ of all K-quasiregular mappings of B^n into B^n. In a less precise form our theorem is contained in [**R1, R2; R3**, p. 38; **MRV**, 3.1]. For the definition and some basic properties of quasiregular mappings the reader is referred to [**MRV, I, R3**].

1.10. THEOREM. *For $n \geqslant 2$, $r \in (0,1)$, and $K \in [1,\infty)$ there exists a number $a(r)$ with $\lim_{r\to 0} a(r) = 1$ such that if $f \in QR_K(B^n)$ then*

$$|f(x) - f(y)| \leqslant a(r)\lambda_n^{1-\alpha}|x-y|^{\alpha} \leqslant a(r)2^{1-1/K}K|x-y|^{\alpha}$$

for all $x, y \in \overline{B}^n(r)$, where $\alpha = K^{1/(1-n)}$.

We shall actually prove a sharp version of Theorem 1.10, which is perhaps new even for $n = 2$; the above simplified formulation sacrifices the sharpness.

For our third result we require the quasihyperbolic metric k_D of a proper subdomain D of R^n. For $a, b \in D$ we set

$$k_D(a,b) = \inf_{\gamma} \int_{\gamma} d(x,\partial D)^{-1}\,ds \tag{1.11}$$

[GP, GO], where the infimum is taken over all rectifiable arcs γ joining a and b in D. We shall prove the following dimension-free version of a distortion theorem due to Gehring and Osgood [**GO**, Theorem 3].

1.12. THEOREM. *Let $f\colon D \to D'$ be a K-quasiconformal mapping, where D and D' are proper subdomains of R^n. Then*

$$\begin{aligned} k_{D'}(f(x), f(y)) &\leqslant c(K)\max\{k_D(x,y)^{\alpha}, k_D(x,y)\} \\ &\leqslant c(K)\max\{k_D(x,y)^{1/K}, k_D(x,y)\} \end{aligned}$$

for all $x, y \in D$, where $\alpha = K^{1/(1-n)}$ and $c(K)$ is a constant depending only on K.

The proof of this theorem makes use of recent results on Teichmüller's modulus problem in R^n [**Vu3**]. We also obtain lower and upper bounds for the least constant $c(K)$ for which this theorem holds.

2. The extremal distortion function $\varphi^*_{K,n}(r)$. We shall adopt the relatively standard notation and terminology of [**V**]. Unit vectors in the directions of the rectangular coordinate axes in R^n are denoted by $e_1, \ldots, e_n$. For $x \in R^n$ and $r > 0$ we let $B^n(x,r) = \{z \in R^n\colon |x-z| < r\}$, $S^{n-1}(x,r) = \partial B^n(x,r)$, $B^n(r) = B^n(0,r)$, $S^{n-1}(r) = \partial B^n(r)$, $B^n = B^n(1)$, and $S^{n-1} = \partial B^n$.

The set of Möbius transformations in $\overline{R}^n = R^n \cup \{\infty\}$ is denoted by $GM(\overline{R}^n)$, and the subset of sense-preserving transformations by $M(\overline{R}^n)$. For nonempty $D \subset \overline{R}^n$ we let $GM(D) = \{f \in GM(\overline{R}^n)\colon fD = D\}$ and $M(D) = \{f \in M(\overline{R}^n)\colon fD = D\}$. We shall require the Poincaré metric $\rho(x,y)$ on B^n defined by

$$\tanh^2\frac{\rho(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2 + (1-|x|^2)(1-|y|^2)} \tag{2.1}$$

for $x, y \in B^n$ or, equivalently, by

$$\sinh^2\frac{\rho(x,y)}{2} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)} \tag{2.2}$$

for $x, y \in B^n$ (cf. [**B**, p. 40]). In particular,

$$\rho(0, x) = \log\frac{1+|x|}{1-|x|} = 2\operatorname{arctanh}|x|$$

for $x \in B^n$. It is a basic fact that ρ is a $GM(B^n)$-invariant function—that is, $\rho(x, y) = \rho(Tx, Ty)$ for all $T \in GM(B^n)$ and all $x, y \in B^n$.

The results in this section rely essentially on the following lemma, which is a sharpened version of [**Vu2**, 3.5]. By means of (1.1) one can show that this lemma is equivalent to the Schwarz lemma (1.4) if $n = 2$, while for $n \geqslant 3$ a different result is obtained (see (2.15) below).

2.3. LEMMA. *Let $K \geqslant 1$, $n \geqslant 2$, and $f \in QC_K(B^n)$. Then*

$$\sinh^2\frac{\rho'}{2} \leqslant \tau^{-1}\left(\frac{1}{K}\tau\left(\sinh^2\frac{\rho}{2}\right)\right) \tag{2.4}$$

and

$$\tanh\frac{\rho'}{4} \leqslant 2\left(\tanh\frac{\rho}{4}\right)^{1/K}, \tag{2.5}$$

where $\rho = \rho(x, y)$ and $\rho' = \rho(f(x), f(y))$ for all $x, y \in B^n$.

PROOF. The conformal invariant $\lambda_{B^n}(x, y)$ examined in [**Vu2**] has the following explicit expression (see [**Vu2**, 2.23]):

$$\lambda_{B^n}(x, y) = \frac{1}{2}\tau_n\left(\sinh^2\frac{\rho(x, y)}{2}\right). \tag{2.6}$$

The proof of (2.4) now follows directly from (2.6) and [**Vu2**, Lemma 3.1(2), 2.20]. It follows from (2.6) and [**Vu2**, 2.14(2)] that

$$\lambda_{B^n}(x, y) \geqslant c_n \log(\coth \rho(x, y)/4),$$

while

$$\lambda_{fB^n}(f(x), f(y)) \leqslant \lambda_{B^n}(f(x), f(y)) \leqslant \frac{c_n}{2}\log\left(4\coth^2\frac{\rho(f(x), f(y))}{4}\right)$$

by (2.6) and [**Vu2**, (5.23)]. Here c_n is a constant depending only on n [**V**, 10.9]. These two inequalities together with [**Vu2**, 3.1(2)] yield inequality (2.5). □

2.7. THEOREM. *If $f \in QC_K(B^n)$ and $f(0) = 0$, then*

$$|f(x)|/\left(1 + \sqrt{1 - |f(x)|^2}\right) \leqslant 2\left(|x|/\left(1 + \sqrt{1 - |x|^2}\right)\right)^{1/K}$$

for all $x \in B^n$.

PROOF. The proof follows immediately from (2.5) and the observation that

$$\tanh\left(\frac{1}{4}\log\frac{1+s}{1-s}\right) = \frac{s}{1+\sqrt{1-s^2}}$$

for $s \in [0, 1)$. □

2.8. Corollary. *For all* $n \geqslant 2$, $K \geqslant 1$, *and* $r \in [0,1]$,

$$\varphi^*_{K,n}(r) \leqslant \min\{2, K\} 2^{1-1/K} r^{1/K}. \tag{2.9}$$

Proof. Let $f \in QC_K(B^n)$ with $f(0) = 0$, and $|x| = r < 1$. Since there is nothing to prove if $|f(x)| \leqslant |x| = r$, we assume that $|f(x)| > |x|$. Then by Theorem 2.7

$$|f(x)| \leqslant 2 \frac{1 + \left(1 - |f(x)|^2\right)^{1/2}}{\left(1 + \left(1 - |x|^2\right)^{1/2}\right)^{1/K}} |x|^{1/K} \leqslant 2\left(1 + \left(1 - |x|^2\right)^{1/2}\right)^{1-1/K} |x|^{1/K}$$
$$\leqslant 2 \cdot 2^{1-1/K} |x|^{1/K}.$$

From definition (1.3) it now follows that $\varphi^*_{K,n}(r) \leqslant 2^{2-1/K} r^{1/K}$. Finally, since $\varphi_{K,n}(r) \leqslant K \cdot 2^{1-1/K} r^{1/K}$ and $\varphi^*_{K,n}(r) \leqslant \varphi_{K,n}(r)$ by (1.6) and (1.4), we obtain (2.9). □

2.10. Corollary. *For* $n \geqslant 2$, $K \geqslant 1$, *and* $r \in [0,1]$, *the extremal distortion function* $\varphi^*_{K,n}$ *satisfies*

$$\varphi^*_{K,n}(r) \leqslant 4^{1-1/K^2} r^{1/K} \tag{2.11}$$

and

$$\varphi^*_{K,n}(r) \leqslant 8^{1-1/K} r^{1/K}. \tag{2.12}$$

Furthermore, the constant 4 *in* (2.11) *cannot be replaced by a smaller absolute constant.*

Proof. One can derive both inequalities from (2.9) by elementary methods, considering separately the two cases $1 \leqslant K \leqslant 2$ and $K \geqslant 2$. We shall give details only for (2.11).

If $1 \leqslant K \leqslant 2$, then by (2.9) it suffices to prove that $4^{1-1/K^2} \geqslant 2^{1-1/K} K$ or, equivalently, that

$$2^{1-2/K^2+1/K} \geqslant K. \tag{2.13}$$

Let $g(K) = (1 - 2K^{-2} + K^{-1})\log 2 - \log K$. Then $g(1) = g(2) = 0$ and $g''(K) < 0$ for $1 \leqslant K \leqslant 2$; (2.13) holds accordingly.

In case $K \geqslant 2$, (2.11) follows from (2.9) and the easily verified fact that $4^{1-1/K^2} \geqslant 2^{1-1/K} \cdot 2$ for such K.

To prove that 4 is the best constant in (2.11) we consider the case $n = 2$. Suppose that $\varphi_{K,2}(r) \leqslant d^{1-1/K^2} r^{1/K}$ for all $K \geqslant 1$ and all $r \in [0,1]$. Then by [**LV**, p. 65]

$$4^{1-1/K} = \lim_{r \to 0} r^{-1/K} \varphi_{K,2}(r) \leqslant d^{1-1/K^2}.$$

Letting $K \to \infty$ yields $d \geqslant 4$ as desired. □

The proof of Theorem 1.8 follows from (2.11) and the fact that $\varphi^*_{1,n}(r) = r$. It should be observed that for $n \geqslant 3$ and $K > 1$, the exponent $1/K$ in (2.11) is not equal to the best exponent $\alpha = K^{1/(1-n)}$ (see (1.4) and (1.6)). On the other hand, the upper bounds in Corollary 2.10 are bounded as K tends to ∞, while the constant $2^{1-1/K} K$ in (1.6) fails to have this property.

2.14. REMARK. For a function $f \in QC_K(B^n)$ with $f(0) = 0$ one obtains from (2.11), (1.4), and (1.6) the following improved variant of the Schwarz lemma.

$$(2.15) \qquad |f(x)| \leqslant \min\left\{2^{1-1/K}K|x|^{\alpha}, 4^{1-1/K^2}|x|^{1/K}\right\}$$

for all $x \in B^n$. By exploiting inequality (2.15) one can slightly improve the functions η and Θ constructed in [**AVV**, (5.22) and **AVV**, Theorem 6.2], respectively.

2.16. DEFINITION. For $n \geqslant 2$, $K \geqslant 1$, we set

$$(2.17) \quad \varphi^*_{1/K,n}(r) = \inf\left\{|f(x)|: f \in QC_K(B^n),\, f(0) = 0,\, |x| = r,\, fB^n = B^n\right\}$$

for $r \in [0, 1)$ and $\varphi^*_{1/K,n}(1) = 1$, extending definition (1.3).

2.18. REMARK. If $K \geqslant 1$ and $n \geqslant 2$ then, by (1.4) and the proof of [**AVV**, Theorem 4.10],

$$\varphi_{1/K,n}(r) \leqslant \varphi^*_{1/K,n}(r) \leqslant r^{\beta} \leqslant r^{\alpha} \leqslant \varphi^*_{K,n}(r) \leqslant \varphi_{K,n}(r),$$

where $\alpha = K^{1/(1-n)} = 1/\beta$. Moreover, by the proof of [**AVV**, Theorem 4.9],

$$\varphi^*_{1/K,n}(r) \leqslant \varphi_{\alpha,2}(r) \leqslant \varphi_{\beta,2}(r) \leqslant \varphi^*_{K,n}(r)$$

(cf. also (1.5)).

2.19. REMARK. It is known that $\varphi^*_{K,2} = \varphi_{K,2}$ for $K \geqslant 1$ (cf. (1.5)). One may also show that $\varphi^*_{1/K,2} = \varphi_{1/K,2}$ for $K \geqslant 1$ by considering the inverse of the Grötzsch extremal K-quasiconformal mapping of B^2 onto itself (cf. [**LV**, Theorem 3.1, p. 64]).

2.20. LEMMA. *For* $n \geqslant 2$, $K \geqslant 1$, *and* $r \in [0, 1]$,

$$\varphi^*_{1/K,n}(r) \geqslant 2^{1-K}K^{-K}r^{\beta},$$

where $\beta = K^{1/(n-1)} = 1/\alpha$.

PROOF. By [**AVV**, Theorem 4.2 and (4.12)] we see that $\varphi^*_{1/K,n}(r) \geqslant \varphi_{1/K,n}(r) \geqslant \lambda_n^{1-\beta}r^{\beta}$. But from the proof of [**AVV**, Corollary 4.14] and the fact that $1 - \alpha = (\beta - 1)\alpha$ it follows that $\lambda_n^{\beta-1} \leqslant 2^{(1-1/K)/\alpha}K^{1/\alpha} \leqslant 2^{K(1-1/K)}K^K = 2^{K-1}K^K$. Thus $\lambda_n^{1-\beta} \geqslant 2^{1-K}K^{-K}$. □

2.21. THEOREM. *Let f be a K-quasiconformal mapping of* B^n *onto* B^n *with* $f(0) = 0$. *Then* $|f(x)| \geqslant 2^{1-2K}|x|^K$ *for all* $x \in B^n$.

PROOF. First, if $|f(x)| \geqslant |x|$, there is nothing to prove. Next, suppose $|f(x)| < |x|$. Since $fB^n = B^n$, Theorem 2.7 applied to f^{-1} yields

$$|x|/\left(1 + \sqrt{1 - |x|^2}\right) \leqslant 2\left(|f(x)|/\left(1 + \sqrt{1 - |f(x)|^2}\right)\right)^{1/K}.$$

Solving this inequality for $|f(x)|$ and making some obvious estimates gives

$$\begin{aligned} |f(x)| &\geqslant \frac{1 + \sqrt{1 - |f(x)|^2}}{\left(1 + \sqrt{1 - |x|^2}\right)^K}\left(\frac{|x|}{2}\right)^K \\ &\geqslant \left(1 + \sqrt{1 - |x|^2}\right)^{1-K}\left(\frac{|x|}{2}\right)^K \geqslant 2^{1-2K}|x|^K, \end{aligned}$$

as desired. □

2.22. COROLLARY. *For* $n \geqslant 2$, $K \geqslant 1$, *and* $r \in [0,1]$,

$$\varphi^{*}_{1/K,n}(r) \geqslant \max\{2^{1-K}K^{-K}r^{\beta}, 2^{1-2K}r^{K}\}.$$

PROOF. This is clear by Lemma 2.20 and Theorem 2.21. □

2.23. COROLLARY. *For* $n \geqslant 2$, $K \geqslant 1$, *and* $r \in [0,1]$,

$$\varphi^{*}_{1/K,n}(r) \geqslant 4^{1-K^2}r^{\beta}.$$

PROOF. By Corollary 2.22 it suffices to show that

$$2^{1-K}K^{-K} \geqslant 4^{1-K^2} \quad \text{for } K \geqslant 1.$$

It is easy to see that this is equivalent to

$$h(K) \equiv K\log K + (1 + K - 2K^2)\log 2 \leqslant 0$$

for $K \geqslant 1$. But since $h(1) = 0$, $h'(1) = 1 - 3\log 2 < 0$, and $h''(K) = 1/K - 4\log 2 \leqslant 1 - 4\log 2 < 0$ we have $h'(K) < 0$ for $K \geqslant 1$. Hence h is decreasing, so that $h(K) \leqslant 0$ for all $K \geqslant 1$. □

2.24. THEOREM. *For* $n \geqslant 2$, $K \geqslant 1$, *and* $r \in [0,1]$,

$$\varphi^{*2}_{K,n}(r) + \varphi^{2}_{1/K,n}(r') \leqslant 1 \tag{2.25}$$

and

$$\varphi^{*2}_{1/K,n}(r) + \varphi^{2}_{K,n}(r') \geqslant 1, \tag{2.26}$$

where $r' = \sqrt{1 - r^2}$. *In each case there is equality for all* r *when* $n = 2$ *or* $K = 1$.

PROOF. We prove only (2.25), since the proof of (2.26) is similar. By Lemma 2.3, $\tau_n(\sinh^2(\rho'/2)) \geqslant \tau_n(\sinh^2(\rho/2))/K$ for $f \in QC_K(B^n)$, $f(0) = 0$, $\rho' = \rho(f(x), f(y))$, and $\rho = \rho(x, y)$. Using the identity $\sinh^2 A = (\tanh^2 A)/(1 - \tanh^2 A)$ and (1.1) we obtain

$$\gamma_n\left((1 - \tanh^2(\rho'/2))^{-1/2}\right) \geqslant (1/K)\gamma_n\left((1 - \tanh^2(\rho/2))^{-1/2}\right).$$

Now set $y = 0 = f(y)$ and $|x| = r$. Since $\rho(x, y) = 2 \operatorname{arctanh} r$ and $\rho(f(x), f(y)) = 2 \operatorname{arctanh} |f(x)|$, we obtain

$$\gamma_n\left(\left(1 - |f(x)|^2\right)^{-1/2}\right) \geqslant (1/K)\gamma_n\left((1 - r^2)^{-1/2}\right). \tag{2.27}$$

Solving (2.27) for $|f(x)|$ gives

$$|f(x)|^2 \leqslant 1 - \varphi^{2}_{1/K,n}(\sqrt{1 - r^2}),$$

from which (2.25) follows when we take the supremum over all such f.

When $n = 2$, equality holds in (2.25) and (2.26) for each $K > 0$ and $r \in [0,1]$. This fact follows immediately from (1.5) and 2.19. □

2.28. REMARK. In view of (1.4) and Sychev's conjecture (1.7) it is of interest to observe that $\varphi_{K,n}(r)$ is not majorized by $4^{1-\alpha}r^{\alpha}$, $\alpha = K^{1/(1-n)}$, for $K > 1$ and large n. This is true because by [AVV, (4.5)]

$$r < \tanh(K \operatorname{arctanh} r) \leqslant \varphi_{K,n}(r)$$

for all $n \geqslant 2$, $K > 1$, and $r \in (0,1)$, while $4^{1-\alpha}r^{\alpha}$ tends to r as n tends to ∞. On the other hand, we do not know whether $\varphi_{K,n}(r)$ is majorized by $4^{1-1/K}r^{1/K}$ for $n \geqslant 3$ and $K > 1$.

3. A sharp bound for $|f(x) - f(y)|$. Several well-known Hölder-continuity theorems already existing in the literature have been cited in the introduction. Although some of these results yield the best Hölder exponent $\alpha = K^{1/(1-n)}$, it seems that very few give the correct limit behavior as $K \to 1$. We shall prove such a theorem, which is suggested by a problem of T. Iwaniec.

For $x \in B^n$ and $M \in (0, \infty)$ we let $D(x, M)$ denote the non-Euclidean ball $\{z \in B^n: \rho(x,z) < M\}$. The Euclidean diameter of a set E in R^n will be denoted by $d(E)$. It follows easily from the well-known formula for $D(z, M)$ [**Ah2**, p. 86, (4), (5); **Vu2**, (2.4)] that

$$d(D(z,M)) \leqslant d(D(0,M)) = 2\tanh(M/2).$$

Next, given $x, y \in B^n$ let $z \in B^n$ be such that $\rho(x,z) = \rho(y,z) = \rho(x,y)/2$. Then

$$|x-y| \leqslant d(D(z,\rho(x,y)/2)) \leqslant 2\tanh(\rho(x,y)/4) \tag{3.1}$$

for all $x, y \in B^n$.

Moreover, by [**Ah2**, (32), p. 27],

$$|x-y| = [x,y]\tanh\frac{\rho(x,y)}{2} \geqslant (1-|x|\,|y|)\tanh\frac{\rho(x,y)}{2} \tag{3.2}$$

for all $x, y \in B^n$, where $[x,y]^2 = |x|^2|y|^2 - 2x\cdot y + 1$.

In summary, we have

3.3. LEMMA. *If $T \in GM(B^n)$ and $x, y \in B^n$, then*

$$(1-|Tx|\,|Ty|)\tanh(\rho(x,y)/2) \leqslant |Tx - Ty| \leqslant 2\tanh(\rho(x,y)/4).$$

The first bound is sharp if $Ty = 0$; the second is sharp if T is a rotation fixing the origin and $x = -y$.

PROOF. Because T is an isometry in the Poincaré metric ρ the result follows from (3.1) and (3.2). □

The main result of this section, perhaps new even for $n = 2$, is the following generalization of Lemma 3.3.

3.4. THEOREM. *Let f be a K-quasiregular mapping of B^n into B^n. Then*

$$|f(x) - f(y)| \leqslant b_K(\tanh\tfrac{1}{2}\rho(x,y)) \tag{3.5}$$

for all $x, y \in B^n$, where $b_K(s) = 2\varphi_{K,n}(s)/(1+\sqrt{1-\varphi_{K,n}^2(s)})$. The result is sharp if f is a rotation fixing the origin and $x = -y$.

PROOF. If we let $t' = \frac{1}{2}\rho(f(x), f(y))$, it follows from (3.1) that

$$|f(x) - f(y)| \leqslant 2\tanh\frac{t'}{2} = \frac{2\tanh t'}{1+\sqrt{1-\tanh^2 t'}}. \tag{3.6}$$

Since $\tanh t' \leqslant \varphi_{K,n}(\tanh\frac{1}{2}\rho(x,y))$ by [**MRV**, 3.1] (cf. [**Vu2**, 3.3]) and since the right side of (3.6) is an increasing function of t', we obtain (3.5). Since $\varphi_{1,n}(r) \equiv r$, the sharpness assertion follows from Lemma 3.3. □

3.7. COROLLARY. *Under the hypotheses of Theorem* 3.4,

$$|f(x) - f(y)| \leqslant (2\lambda_n)^{1-\alpha}\rho(x,y)^\alpha, \qquad \alpha = K^{1/(1-n)}, \tag{3.8}$$

and

$$|f(x) - f(y)| \leqslant \varphi_{K,n}(a) + \varphi_{K,n}^2(a), \qquad a = \tanh\frac{\rho(x,y)}{2}, \tag{3.9}$$

for all $x, y \in B^n$.

PROOF. It follows from (3.1) and (1.6) (cf. [**Vu2**, 3.3(1)]) that

$$\begin{aligned}|f(x) - f(y)| &\leqslant 2\tanh(\rho(f(x), f(y))/2)\\ &\leqslant 2\varphi_{K,n}(\tanh(\rho(x,y)/2)) \leqslant 2\lambda_n^{1-\alpha}(\rho(x,y)/2)^\alpha\end{aligned}$$

and (3.8) is proved. Formula (3.9) follows from (3.5) and the second part of the elementary inequality

$$\frac{4(1+x)}{3+2\sqrt{2}} \leqslant \frac{2}{1+\sqrt{1-x^2}} \leqslant 1 + x \tag{3.10}$$

for $0 \leqslant x \leqslant 1$. □

A drawback of the sharp inequality (3.5) is its relatively complicated form. In order to derive a more practical alternative form we recall some facts about elliptic integrals. As in [**LV**, p. 60] we let

$$\mu(r) = \frac{\pi\mathcal{K}(\sqrt{1-r^2})}{2\mathcal{K}(r)}, \qquad \mathcal{K}(r) = \int_0^1 [(1-x^2)(1-r^2x^2)]^{-1/2}\,dx,$$

for each $r \in (0,1)$. We shall need the well-known inequality

$$\log(1/r) < \mu(r) < \log(4/r), \qquad 0 < r < 1 \tag{3.11}$$

(cf. [**LV**, p. 61]).

3.12. LEMMA. *For* $n \geqslant 2$, $K \geqslant 1$, *and* $0 < r < 1$, *the following inequalities hold*:

$$\frac{1+t^{2K}}{(1+t^K)^2} \leqslant \frac{1}{1+\sqrt{1-\varphi_{K,n}^2(r)}} \leqslant \frac{1}{2}(1 + \varphi_{K,n}(r)), \tag{3.13}$$

where $t = ((1-r)/(1+r))^{1/2}$. *Moreover,*

$$\frac{1}{1+\sqrt{1-\varphi_{K,n}^2(r)}} \leqslant \frac{1+(t/2)^{2K}}{\left(1+(t/2)^K\right)^2}. \tag{3.14}$$

PROOF. By [**AVV**, (4.5)],

$$\frac{(1+r)^K - (1-r)^K}{(1+r)^K + (1-r)^K} \leqslant \varphi_{K,n}(r) \leqslant \tanh\left(\frac{K}{2}\mu\left(\frac{1-r}{1+r}\right)\right) \tag{3.15}$$

for $K \geqslant 1$. The lower bound in (3.13) follows from the lower bound in (3.15). The upper bound in (3.13) is a consequence of the second inequality in (3.10). Finally, from the upper bounds in (3.15) and (3.11) we see that

$$\varphi_{K,n}(r) \leqslant \tanh\left(K \log \frac{2}{t}\right) = \frac{1-(t/2)^{2K}}{1+(t/2)^{2K}},$$

and (3.14) follows by an easy computation. □

3.16. Remark. For very small values of r, say for $r \in (0, r_0)$, where $\lambda_n^{1-\alpha} r_0^\alpha = 1/2$ and $\alpha = K^{1/(1-n)}$, one can use inequality (1.6) instead of [**AVV**, (4.5)] to prove an estimate for $\varphi_{K,n}$ that is slightly different from those in Lemma 3.12.

3.17. Proof of Theorem 1.10. Let $r \in (0,1)$ and $x, y \in \overline{B}^n(r)$. By (3.5), (1.6), (2.1), and (3.13),

$$\begin{aligned} |f(x)-f(y)| &\leqslant b_K\left(\tanh \frac{\rho(x,y)}{2}\right) \\ &\leqslant \frac{2\lambda_n^{1-\alpha}(\tanh(\rho(x,y)/2))^\alpha}{1+\sqrt{1-\varphi_{K,n}^2(2r/(1+r^2))}} \\ &\leqslant \frac{\lambda_n^{1-\alpha}(1+\varphi_{K,n}(2r/(1+r^2)))|x-y|^\alpha}{\left[|x-y|^2+(1-|x|^2)(1-|y|^2)\right]^{\alpha/2}}. \end{aligned}$$

From these estimates and from [**AVV**, Theorem 4.4] it follows that we may choose

$$\begin{aligned} a(r) &= \left(1+\varphi_{K,n}\left(\frac{2r}{1+r^2}\right)\right)(1-r^2)^{-\alpha} \\ &\leqslant \left(1+\tanh\left(\frac{K}{2}\mu\left(\left(\frac{1-r}{1+r}\right)^2\right)\right)\right)(1-r^2)^{-\alpha} \end{aligned}$$

in Theorem 1.10. Inequality (3.11) implies that $\lim_{r\to 0} a(r) = 1$. Finally, $\lambda_n^{1-\alpha} \leqslant 2^{1-1/K}K$ by [**AVV**, 4.14]. □

3.18. Remarks. (1) If we set $K = 1$ and $n = 2$ in (3.5) we find that the upper bound in Lemma 3.3 holds not just for $GM(B^n)$ but for all analytic functions f: $B^2 \to B^2$.

(2) C. Carathéodory [**Ca**, Satz 1] proved that

$$|f(x)-f(y)|/|x-y| \leqslant 1$$

for distinct points $x, y \in \overline{B}^2(\sqrt{2}-1)$ when f: $B^2 \to B^2$ is an analytic function with $f(0) = 0$. His result is closely related to Theorem 1.10, although no normalization at 0 is required in 1.10.

(3) For $f \in M(B^n)$ one may improve Theorem 1.10 in the following way. By conformal invariance of the hyperbolic metric, (2.2) yields the estimate

$$|f(x)-f(y)| \leqslant \frac{1}{1-r^2}|x-y|$$

for all $x, y \in \overline{B}^n(r)$, $r \in (0,1)$.

4. Distortion of the quasihyperbolic metric. Let D, D' be domains in R^n, $L \geqslant 1$, and $f\colon D \to D'$ a homeomorphism such that

$$|x-y|/L \leqslant |f(x)-f(y)| \leqslant L|x-y| \tag{4.1}$$

for all $x, y \in D$. Then we say that f is *L-bi-Lipschitz*. If (4.1) holds with $L = 1$ then f is called a *Euclidean isometry*.

4.2. LEMMA. *Let D, D' be proper subdomains of R^n and let $f\colon D \to D'$ be an L-bi-Lipschitz homeomorphism. Then*

$$L^{-2}k_D(a,b) \leqslant k_{D'}(f(a),f(b)) \leqslant L^2 k_D(a,b) \tag{4.3}$$

for all $a, b \in D$, where k_D is the quasihyperbolic metric defined in (1.11). *In particular, k_D is invariant under Euclidean isometries.*

PROOF. It is enough to prove the second inequality, since the first follows from application of the second to f^{-1}.

Suppose γ is a rectifiable arc in D joining a to b. By Theorem 5.3 of [**V**, p. 12] we have

$$\int_{f\circ\gamma} d(y,\partial D')^{-1}|dy| \leqslant L\int_\gamma d(f(x),\partial D')^{-1}|dx|. \tag{4.4}$$

Choose $\zeta' \in \partial D'$ such that $d(f(x),\partial D') = |f(x)-\zeta'|$. Then there exists a sequence $y_m \in D'$ such that $\lim y_m = \zeta'$. Let $x_m = f^{-1}(y_m)$. By choosing a subsequence, we may assume that $x_m \to \zeta \in \partial D$. Then

$$|f(x)-y_m| = |f(x)-f(x_m)| \geqslant L^{-1}|x-x_m|. \tag{4.5}$$

Hence letting m tend to ∞ we get $d(f(x),\partial D')^{-1} \leqslant Ld(x,\partial D)^{-1}$ for all $x \in \gamma$. The second inequality in (4.3) follows if we combine (4.4) and (4.5), then take the infimum over all γ joining a and b in D. □

We now define

$$j_D(x,y) = \log(1+|x-y|/\min\{d(x),d(y)\}),$$

where $d(z) = d(z,\partial D)$ for $z \in D$ (cf. [**GO**; **Vu2**, (2.26)]).

4.6. COROLLARY. *Under the hypotheses of Lemma* 4.2, *we have*

$$L^{-2}j_D(x,y) \leqslant j_{D'}(f(x),f(y)) \leqslant L^2 j_D(x,y)$$

for all $x, y \in D$.

PROOF. As in Lemma 4.2, it is enough to prove the second inequality. We may assume that $d(f(x),\partial D') \leqslant d(f(y),\partial D')$. Then

$$\begin{aligned} j_{D'}(f(x),f(y)) &= \log(1+|f(x)-f(y)|/d(f(x),\partial D')) \\ &\leqslant \log(1+L^2|x-y|/d(x,\partial D)) \\ &\leqslant L^2\log(1+|x-y|/d(x,\partial D)) \leqslant L^2 j_D(x,y). \end{aligned}$$

Here we have used the proof of Lemma 4.2 and Bernoulli's inequality $t\log(1+x) \geqslant \log(1+tx)$ for $x \geqslant 0$ and $t \geqslant 1$ (cf. [**Vu1**, (2.33)]). □

The next lemma generalizes [**GO**, Lemma 2].

4.7. LEMMA. *For each* $K \geqslant 1$ *there exists a constant* $a_1 \in ((\frac{1}{2}(\sqrt{3} - \sqrt{2}))^{8K}, 2(\sqrt{3} - \sqrt{2})^{8K})$ *with the following property. If* D, D' *are proper subdomains of* R^n, $n \geqslant 2$, *and if* $f\colon D \to D'$ *is K-quasiconformal, then*

$$|f(x) - f(y)|/d(f(x), \partial D') \leqslant 1/2$$

and

$$\frac{|f(x) - f(y)|}{d(f(x), \partial D')} \leqslant \frac{1}{2}\varphi^*_{K,n}\left(\frac{|x-y|}{a_1 d(x,\partial D)}\right) \leqslant \frac{K}{2^{1/K}a_1}\left(\frac{|x-y|}{d(x,\partial D)}\right)^{\alpha}$$

for $y \in B^n(x, a_1 d(x, \partial D))$.

PROOF. The second assertion follows easily from the first one and from definitions (1.3), (1.4), and (1.6). Hence it suffices to prove the first assertion. Since only the maximal inscribed ball $B^n(x, d(x, \partial D))$ will matter in the proof, we may assume that $D = B^n$ and $x = 0$. Let

$$r_{D'}(u, v) = \frac{|u - v|}{\min\{d(u, \partial D'), d(v, \partial D')\}}$$

for $u, v \in D'$. Then by **[Vu3**, Theorem 4.5],

$$\lambda_{D'}(f(x), f(y)) \leqslant 4\tau(r_{D'}(f(x), f(y))), \tag{4.8}$$

where $\lambda_{D'}$ represents a conformal invariant studied in **[Vu2]**. By (4.8), **[Vu2**, Lemma 3.1(2)], (2.6), and (2.2), we obtain

$$r_{D'}(f(0), f(y)) \leqslant \tau^{-1}\left(\frac{1}{8K}\tau\left(\frac{|y|^2}{1 - |y|^2}\right)\right).$$

Next, if we require that

$$\tau^{-1}\left(\frac{1}{8K}\tau\left(\frac{|y|^2}{1 - |y|^2}\right)\right) \leqslant \frac{1}{2}$$

and solve for $|y|$, we obtain the upper bound

$$|y|^2 \leqslant \tau^{-1}(8K\tau(\tfrac{1}{2}))/(1 + \tau^{-1}(8K\tau(\tfrac{1}{2}))).$$

Thus we may choose a_1 as $[\tau^{-1}(8K\tau(\frac{1}{2}))/(1 + \tau^{-1}(8K\tau(\frac{1}{2})))]^{1/2}$ in the theorem or, in light of (1.1) and (1.2),

$$a_1 = \sqrt{1 - \varphi^2_{8K,n}(\sqrt{2/3})}\,. \tag{4.9}$$

Finally, from (3.13) and (3.14), with $r = \sqrt{2/3}$ and with K replaced by $8K$, we obtain

$$\frac{2(t/2)^{8K}}{1 + (t/2)^{16K}} \leqslant a_1 = \sqrt{1 - \varphi^2_{8K,n}(r)} \leqslant \frac{2t^{8K}}{1 + t^{16K}},$$

with $t = \sqrt{3} - \sqrt{2}$. Thus

$$\left(\tfrac{1}{2}(\sqrt{3} - \sqrt{2})\right)^{8K} < a_1 < 2(\sqrt{3} - \sqrt{2})^{8K}. \quad \square$$

4.10. LEMMA. *If D is a proper subdomain of R^n then*

$$(4.11)\qquad k_D(x,y) \geqslant j_D(x,y)$$

for all $x, y \in D$ and

$$(4.12)\qquad k_D(x,y) \leqslant \log\big(1 + |x-y|/\big(d(x) - |x-y|\big)\big)$$

for $|x-y| < d(x)$, where $d(x) = d(x, \partial D)$.

PROOF. Inequality (4.11) is **[GP**, (2.2)], and (4.12) is **[Vu1**, Lemma 2.11]. □

4.13. PROOF OF THEOREM 1.12. In **[GO**, Theorem 3] this result was proved with the constant $c = 4(4\lambda_n^2)^{1/\alpha}$, which tends to ∞ as n tends to ∞ **[An]**. We need to show that we can replace this with a universal constant c that does not depend on n. It suffices to prove the first inequality in Theorem 1.12 since

$$\max\{k_D(x,y), k_D(x,y)^\alpha\} = k_D(x,y)^\alpha \leqslant k_D(x,y)^{1/K}$$

for $k_D(x,y) \leqslant 1$, and

$$\max\{k_D(x,y), k_D(x,y)^\alpha\} = k_D(x,y) = \max\{k_D(x,y), k_D(x,y)^{1/K}\}$$

for $k_D(x,y) \geqslant 1$.

Suppose first that

$$(4.14)\qquad |x-y| \leqslant a_1 d(x, \partial D),$$

where a_1 is as in Lemma 4.7. Then by Lemma 4.7,

$$(4.15)\qquad |f(x) - f(y)|/d\big(f(x), \partial D'\big) \leqslant 1/2$$

and hence by (4.12)

$$(4.16)\qquad k_{D'}(f(x), f(y)) \leqslant \log\left(1 + \frac{|f(x)-f(y)|}{d\big(f(x),\partial D'\big) - |f(x)-f(y)|}\right) \leqslant \frac{2|f(x)-f(y)|}{d\big(f(x),\partial D'\big)} \leqslant 1$$

since $\log(1+t) \leqslant t$ for $t \geqslant 0$.

On the other hand, since $|x-y|/d(x,\partial D) \leqslant a_1$ and since $(1/t)\log(1+t)$ is decreasing for $t > 0$ we see by (4.11) that

$$(4.17)\qquad k_D(x,y) \geqslant \log\left(1 + \frac{|x-y|}{d(x,\partial D)}\right) \geqslant \frac{\log(1+a_1)}{a_1}\,\frac{|x-y|}{d(x,\partial D)}.$$

Then by Lemma 4.7 we obtain

$$(4.18)\qquad |f(x)-f(y)| \leqslant \tfrac{1}{2} d\big(f(x),\partial D'\big)\varphi^*_{K,n}\big(|x-y|/\big(a_1 d(x,\partial D)\big)\big).$$

Thus by (4.16), (4.18), (1.4), (1.6), and (4.17) we have

$$(4.19)\qquad k_{D'}(f(x), f(y)) \leqslant \varphi^*_{K,n}\left(\frac{|x-y|}{a_1 d(x,\partial D)}\right) \leqslant 2^{1-1/K}K\left(\frac{|x-y|}{a_1 d(x,\partial D)}\right)^\alpha \leqslant 2^{1-1/K}K\left(\frac{k_D(x,y)}{\log(1+a_1)}\right)^\alpha.$$

Next suppose that

$$|x - y| > a_1 d(x, \partial D) \tag{4.20}$$

and choose $z_1, \ldots, z_{m+1}$ on a quasihyperbolic geodesic in D joining x and y [**GO**] so that $z_1 = x$, $z_{m+1} = y$, and

$$|z_l - z_{l+1}|/d(z_l, \partial D) = a_1, \qquad |z_m - z_{m+1}|/d(z_m, \partial D) \leqslant a_1$$

for $l = 1, \ldots, m-1$. Then by (4.11)

$$k_D(x, y) \geqslant \sum_{l=1}^{m-1} k_D(z_l, z_{l+1}) \geqslant (m-1)\log(1 + a_1). \tag{4.21}$$

Hence

$$m \leqslant 1 + k_D(x, y)/\log(1 + a_1). \tag{4.22}$$

Then by (4.12), the definition of the number a_1, and (4.22),

$$\begin{aligned} k_{D'}(f(x), f(y)) &\leqslant \sum_{l=1}^{m} k_{D'}(f(z_l), f(z_{l+1})) \\ &\leqslant \sum_{l=1}^{m} \log\left(1 + \frac{|f(z_l) - f(z_{l+1})|}{d(f(z_l), \partial D') - |f(z_l) - f(z_{l+1})|}\right) \\ &\leqslant \left(1 + \frac{k_D(x, y)}{\log(1 + a_1)}\right)\log 2. \end{aligned} \tag{4.23}$$

Since $k_D(x, y) \geqslant j_D(x, y) \geqslant \log(1 + a_1)$, (4.23) now yields

$$k_{D'}(f(x), f(y)) \leqslant 2(\log 2) k_D(x, y)/\log(1 + a_1), \tag{4.24}$$

provided (4.20) holds.

Finally by (4.19) and (4.24) we obtain, for all $x, y \in D$,

$$\begin{aligned} k_{D'}(f(x), f(y)) &\leqslant \max\{b_1 k_D(x, y)^\alpha, b_2 k_D(x, y)\} \\ &\leqslant \max\{b_1, b_2\} \max\{k_D(x, y)^\alpha, k_D(x, y)\}, \end{aligned} \tag{4.25}$$

where $b_1 = 2^{1-1/K} K(\log(1 + a_1))^{-\alpha}$ and $b_2 = (2\log 2)/\log(1 + a_1)$. From Lemma 4.7 it follows easily that

$$\max\{b_1, b_2\} \leqslant 2K/\log(1 + a_1).$$

Therefore, by (4.25), we may choose $c(K) = 2K/\log(1 + a_1)$ in Theorem 1.12. □

Next, we define

$$\bar{c}(K) = \inf\{c(K)\colon \text{Theorem 1.12 holds with } c(K)\}. \tag{4.26}$$

We show that $\bar{c}(K) \to \infty$ as $K \to \infty$ and provide quantitative lower and upper bounds. To this end the following two lemmas are needed.

4.27. LEMMA. *Let $G = R^n \setminus \{0\}$, let $x, y \in G$, and let φ be the angle determined by the segments* $[0, x]$ *and* $[0, y]$, $\varphi \in [0, \pi]$. *Then*

$$k_G(x, y) = \sqrt{\varphi^2 + \log^2(|x|/|y|)}\,.$$

PROOF. This formula is developed in [**MO**, §2]. □

Now let $\lambda(K) = (\mu^{-1}(\pi/2K)/\mu^{-1}(\pi K/2))^2$ denote the well-known distortion coefficient of Lehto, Virtanen, and Väisälä **[LVV]** (cf. **[LV**, pp. 81, 82, 106–108]). In particular, $\lambda(1) = 1$ and $\lambda(K)$ tends to ∞ as K tends to ∞.

4.28. LEMMA. *There exists a K-quasiconformal mapping f: $R^n \to R^n$ such that $f(0) = 0$, $f(\infty) = \infty$, $f(e_1) = e_1$, and $f(-e_1) = -\lambda(K^{1/(n-1)})e_1$.*

PROOF. This follows by rotation of the two-dimensional extremal quasiconformal mapping of **[LVV]** (cf. **[AVV**, Theorem 1.14]). □

4.29. THEOREM. *The constant $\bar{c}(K)$ in* (4.26) *satisfies*

$$[1 + \pi^{-2}\log^2\lambda(K)]^{1/2} \leqslant \bar{c}(K) \leqslant 2K\left[1 + \left(2(\sqrt{3} + \sqrt{2})\right)^{8K}\right]. \tag{4.30}$$

PROOF. For the lower bound fix f as in Lemma 4.28. Then with $D = R^n \setminus \{0\}$, we have by Lemma 4.27 the relations

$$k_D(e_1, -e_1) = \pi$$

and

$$\begin{aligned} k_D(f(e_1), f(-e_1)) &= k_D(e_1, -\lambda(K^{1/(n-1)})e_1) \\ &= \left[\pi^2 + \log^2\lambda(K^{1/(n-1)})\right]^{1/2}. \end{aligned}$$

Thus by the definition of $\bar{c}(K)$ we have

$$\left[\pi^2 + \log^2\lambda(K^{1/(n-1)})\right]^{1/2} \leqslant \pi\bar{c}(K).$$

Choosing $n = 2$ yields the desired lower estimate.

Finally, by the choice of $c(K)$ in the proof of Theorem 1.12, the estimate $\log(1 + a_1) > a_1/(1 + a_1)$, and Lemma 4.7, we have

$$\bar{c}(K) \leqslant \frac{2K}{\log(1 + a_1)} < 2K(1 + 1/a_1) < 2K\left[1 + \left(2(\sqrt{3} + \sqrt{2})\right)^{8K}\right],$$

and the upper estimate follows. □

We next obtain dimension-free versions of distortion results due to Gehring and Osgood **[GO**, Lemmas 2 and 3].

4.31. LEMMA. *If f is a K-quasiconformal mapping of $\overline{R}^n$ with $f(\infty) = \infty$, then*

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} + 1 \leqslant B\left(\left(\frac{|x - y|}{|x - z|}\right)^K + 1\right),$$

where $B = (1/a_1)\max\{2, \frac{1}{2}(\min\{2, K\})^K\}$.

PROOF. Let $D = R^n \setminus \{y\}$, $D' = R^n \setminus \{f(y)\}$. Then $|f(x) - f(y)| = d(f(x), \partial D')$, $|x - y| = d(x, \partial D)$. We may assume that

$$|f(x) - f(z)|/|f(x) - f(y)| \leqslant 2/B,$$

since otherwise the result holds trivially. Then $|f(x) - f(z)| \leqslant a_1 d(f(x), \partial D')$. So by (2.9) and Lemma 4.7 applied to f^{-1} we get

$$\frac{|x-z|}{|x-y|} \leqslant \frac{1}{2}\varphi^*_{K,n}\left(\frac{|f(x)-f(z)|}{a_1|f(x)-f(y)|}\right)$$
$$\leqslant 2^{-1/K}\min\{2, K\}\left(\frac{|f(x)-f(z)|}{a_1|f(x)-f(y)|}\right)^{1/K},$$

or

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leqslant \frac{(\min\{2,K\})^K}{2a_1}\left(\frac{|x-y|}{|x-z|}\right)^K \leqslant B\left(\frac{|x-y|}{|x-z|}\right)^K,$$

and the result follows. □

4.32. COROLLARY. *Theorem* 4 *and Corollary* 3 *in* **[GO]** *hold with constants that do not depend on* n.

PROOF. In [**GO**, Theorem 4] we can choose $c = 2/\alpha = 2\beta \leqslant 2K$ (as indicated in [**GO**]) and $d = \log B \leqslant \log(\max\{2, 2(\min\{2, K\})^K\}/a_1)$. In [**GO**, Corollary 3] we may choose $c_1 = c(K)$, $c_2 = 2\beta \leqslant 2K$, $d_2 = \log B$. Then the proof follows as in [**GO**]. □

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